Lecture 5

6.5: Exponential Growth and Decay

Many things grow or decay at an exponential rate, such as radioactive substances and population.

This type of growth/decay means the quantity changes in proportion to its size.

Mathematically, this is described by:

If $k>0$, this is the law of natural growth, and if $k<0$, this is the law of natural decay. This is a differential equation, the solution to which is:

Given an initial value $y(0)=y_0$, we can find a specific value for C:

Population Growth

Let $P(t)$ denote the population size at time t. Then $P'(t) = kP(t)$, where k is the relative growth rate, i.e., $k = \frac{P'(t)}{P(t)}$. If $P(0) = P_0$ is the initial population, then the model for the population is given by

$$P(t) =$$

Ex: In Japan, there is an island called Ōkunoshima, a.k.a. Rabbit Island. In 1971, 8 rabbits were released onto the island, and, as of 2011, there are about 300 rabbits.

- What is the relative growth rate of the rabbit population?

- What is $P(0) = P_0$?
- What differential equation does $P(t)$ satisfy?
- $P(t) =$
- Approximate rabbit population in 2021?

Radioactive Decay

Let $m(t)$ denote the mass at time t. Then $m'(t) = k\,m(t)$. Since the substance is decaying, $k < 0$, and so the relative decay rate is

$$-k = \frac{-m'(t)}{m(t)}.$$

If $m(0) = m_0$ is the initial mass, the model for the remaining mass is:

$$m(t) =$$

The decay rate of radioactive substances is often described in terms of half-life, the amount of time it take for a substance to lose half of its mass. If we call the half-life of the substance: h, we can recover the familiar formula from chemistry/physics:

Ex: ^{233}Pu has a half-life of 20 minutes. Suppose we have a 100mg sample of ^{233}Pu.

- Find a formula for the mass remaining after t minutes:

- How much remains after one hour?

- How long will it take for there to be 10mg of ^{233}Pu remaining?

Compound Interest

Compounded Continuously

Suppose we invest A_0 dollars into an account with interest rate r, compounded continuously. After t years, the amount in the account is given by:

$$A(t) =$$

Ex: John invests $1000 into an account paying 5% interest, compounded continuously. How much is in the account after 5 years?

How long until the balance is $2000?

Compounded Discretely

The alternative to compounding continuously is to compound n times a year, e.g.,

compounded	annually	semiannually	quarterly	monthly	daily	...
$n =$						

If the account has an interest rate of r, and is compounded n times a year, after t years an initial investment of A_0 dollars will have value

$$A(t) =$$

If we let $n \to \infty$, we see the recovery of the compounded continuously case.

Ex: If \$50,000 is borrowed at 10% interest, compounded quarterly, how much is owed after 4 years?